S185. Let A_1, A_2, A_3 be non-collinear points on parabola $x^2 = 4py$, $p > 0$, and let $B_1 = l_2 \cap l_3, B_2 = l_3 \cap l_1, B_3 = l_1 \cap l_2$ where l_1, l_2, l_3 are tangents to the parabola at points A_1, A_2, A_3, respectively. Prove that $\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]}$ is a constant and find its value.

Proposed by Arkady Alt, San Jose, California, USA

First solution by Evangelos Mouroukos, Agrinio, Greece

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) denote the coordinates of the points A_1, A_2 and A_3 respectively. The equations of the lines ℓ_1, ℓ_2 and ℓ_3 are

$$\ell_1 : xx_1 = 2p(y + y_1),$$

$$\ell_2 : xx_2 = 2p(y + y_2),$$

$$\ell_3 : xx_3 = 2p(y + y_3).$$

Solving the system of ℓ_2 and ℓ_3, we find that the point B_1 has coordinates

$$\left(\frac{2p(y_3 - y_2)}{x_3 - x_2}, \frac{x_2y_3 - x_3y_2}{x_3 - x_2}\right).$$

Since $y_3 - y_2 = \dfrac{x_3^2 - x_2^2}{4p} = \dfrac{(x_3 - x_2)(x_3 + x_2)}{4p}$ and $x_2y_3 - x_3y_2 = \dfrac{x_2x_3^2 - x_3x_2^2}{4p}$, we find that

$$B_1\left(\frac{x_2 + x_3}{2}, \frac{x_2x_3}{4p}\right)$$

and similarly

$$B_2\left(\frac{x_3 + x_1}{2}, \frac{x_3x_1}{4p}\right),$$

$$B_3\left(\frac{x_1 + x_2}{2}, \frac{x_1x_2}{4p}\right).$$

We compute the area of triangle $A_1A_2A_3$:

$$[A_1A_2A_3] = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}\right| =$$

$$= \frac{1}{2}\left|\begin{vmatrix} x_2 - x_1 & \dfrac{(x_2 - x_1)(x_2 + x_1)}{4p} \\ x_3 - x_1 & \dfrac{(x_3 - x_1)(x_3 + x_1)}{4p} \end{vmatrix}\right| = \frac{1}{8p}\left|(x_2 - x_1)(x_3 - x_1)\begin{vmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{vmatrix}\right| =$$

$$= \frac{1}{8p}\left|(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)\right|.$$

A similar calculation yields

$$[B_1B_2B_3] = \frac{1}{2}\left|\begin{vmatrix} \dfrac{x_2 + x_3}{2} & \dfrac{x_2x_3}{4p} & 1 \\ \dfrac{x_3 + x_1}{2} & \dfrac{x_3x_1}{4p} & 1 \\ \dfrac{x_1 + x_2}{2} & \dfrac{x_1x_2}{4p} & 1 \end{vmatrix}\right| = \frac{1}{16p}\left|\begin{vmatrix} x_2 + x_3 & x_2x_3 & 1 \\ x_3 + x_1 & x_3x_1 & 1 \\ x_1 + x_2 & x_1x_2 & 1 \end{vmatrix}\right| =$$

$$= \frac{1}{16p}\left| \begin{vmatrix} x_2+x_3 & x_2x_3 & 1 \\ x_2-x_1 & x_3(x_2-x_1) & 0 \\ x_1-x_3 & x_2(x_1-x_3) & 0 \end{vmatrix} \right| = \frac{1}{16p}\left|(x_2-x_1)(x_1-x_3) \begin{vmatrix} x_2+x_3 & x_2x_3 & 1 \\ 1 & x_3 & 0 \\ 1 & x_2 & 0 \end{vmatrix} \right| =$$

$$= \frac{1}{16p}\left|\,(x_2-x_1)\,(x_3-x_1)\,(x_3-x_2)\,\right|.$$

We conclude that $\boxed{\dfrac{[A_1A_2A_3]}{[B_1B_2B_3]} = 2}$.

Second solution by Daniel Lasaosa, Universidad Pública de Navarra, Spain

Let x_i be the value of x for A_i $(i = 1, 2, 3)$. The slope of $x^2 = 4py$ at $x = x_i$ is clearly $\frac{x_i}{2p}$, or l_i has equation $y = \frac{x_i(2x-x_i)}{4p}$. It follows that B_1 has coordinates satisfying $y = \frac{x_2(2x-x_2)}{4p} = \frac{x_3(2x-x_3)}{4p}$, or $2x(x_2 - x_3) = (x_2 + x_3)(x_2 - x_3)$. Since $x_2 \neq x_3$ (otherwise $A_2 = A_3$), it follows that $x = \frac{x_2+x_3}{2}$ and $y = \frac{x_2x_3}{4p}$ for B_1, and similarly by cyclic permutation for B_2, B_3.

Using the vector product, and since $\overrightarrow{A_1A_i} = \left(x_i - x_1, (x_i - x_1)\frac{x_i+x_1}{4p}\right)$ for $i = 2, 3$, it follows that

$$[A_1A_2A_3] = \frac{|x_1 - x_2||x_2 - x_3||x_3 - x_1|}{8p}.$$

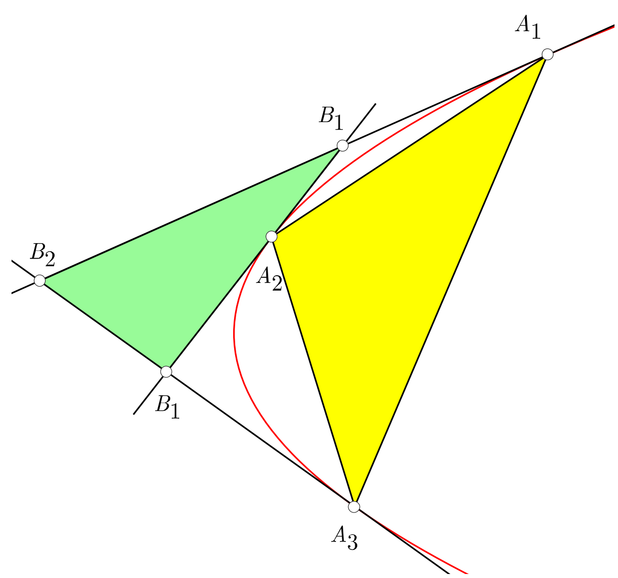
Similarly, and since $\overrightarrow{B_1B_i} = \left(\frac{x_1-x_i}{2}, \frac{x_j(x_1-x_i)}{4p}\right)$, where $\{i, j\} = \{2, 3\}$, it also follows that

$$[B_1B_2B_3] = \frac{|x_1 - x_2||x_2 - x_3||x_3 - x_1|}{16p} = \frac{[A_1A_2A_3]}{2}.$$

The conclusion follows, the proposed ratio is always equal to 2.

Third solution by Ercole Suppa, Teramo, Italy

Let $A_1 = \left(u, \frac{1}{4p}u^2\right)$, $A_2 = \left(v, \frac{1}{4p}v^2\right)$, $A_3 = \left(w, \frac{1}{4p}w^2\right)$.



Observe that the equation of the tangent to the parabola $x^2 = 4py$ at its point $T\left(t, \frac{1}{4p}t^2\right)$ is

$$2t(x - t) - 4p\left(y - \frac{1}{4p}t^2\right) = 0 \qquad \Leftrightarrow \qquad 2tx - 4py - t^2 = 0$$

Therefore the equations of the lines ℓ_1, ℓ_2, ℓ_3 are:

$$\begin{aligned}\ell_1 &: 2ux - 4py - u^2 = 0\\ \ell_2 &: 2vx - 4py - v^2 = 0\\ \ell_1 &: 2wx - 4py - w^2 = 0\end{aligned}$$

After some algebra we get

$$B_1 = \left(\frac{v+w}{2}, \frac{vw}{4p}\right), \qquad B_2 = \left(\frac{u+w}{2}, \frac{uw}{4p}\right), \qquad B_3 = \left(\frac{u+v}{2}, \frac{uv}{4p}\right)$$

$$[A_1A_2A_3] = \frac{1}{2}\left|\det\begin{pmatrix} u & \frac{1}{4p}u^2 & 1\\ v & \frac{1}{4p}v^2 & 1\\ w & \frac{1}{4p}w^2 & 1\end{pmatrix}\right| = \frac{1}{8}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \tag{*}$$

$$[B_1B_2B_3] = \frac{1}{2}\left|\det\begin{pmatrix} \frac{v+w}{2} & \frac{vw}{4p} & 1\\ \frac{u+w}{2} & \frac{uw}{4p} & 1\\ \frac{u+v}{2} & \frac{uv}{4p} & 1\end{pmatrix}\right| = \frac{1}{16}\left|\frac{(u-v)(u-w)(v-w)}{p}\right| \tag{**}$$

Finally, by using (*),(**) we obtain $\frac{[A_1A_2A_3]}{[B_1B_2B_3]} = 2$, establishing the result.

Also solved by Albert Stadler, Switzerland; Roberto Bosch Cabrera, Havana, Cuba; G.R.A.20 Problem Solving Group, Roma, Italy; Daniel Campos Salas, Costa Rica.